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No Arbitrage Pricing of Cross Currency Moving Average Exchange Options

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ABSTRACT

This paper provides an analytic formula of Cross Currency Moving Average Exchange Options, referred as CCMAE Options. It is an option which exchanges one domestic asset for another foreign asset at the average level of price for certain periods of time. Essentially, it is a mixture of quanto, moving average, and exchange options, which are popularly traded options.

We examine two types of CCMAE options with different types of payoff: a fixed foreign exchange rate CCMAE option (Fixed-CCMAE option) and a floating foreign exchange rate CCMAE option (Floating-CCMAE option). We also validate the accuracy of analytic formula of CCMAE options via Monte Carlo simulation, Monte Carlo integration, and numerical integration approaches.

Keywords: cross currency, moving average, exchange options, quanto options

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INTRODUCTION

Recently, globalization and liberalization of financial institutions are drivers to drive domestic investors to invest their money into foreign financial markets. The most famous hedging tool -- Nikkei Put Warrant in the AMEX market, is a qanto-type security, that can be dated back into early 1989. In terms of treatment of foreign exchange rate, a designed security of this type is to fix the foreign exchange rate at the outset of the contract. The result of Nikkei Put Warrants would isolate U.S. investors from directly bearing movements in the foreign exchange rate. To solve the same problems of the risk of foreign exchange rate all around the world, recent studies develop abundant financial tools including: Bennett and Kennedy (2003) pricing the quato by copula functions, Dai et al. (2004) pricing quanto lookback options, Chuang and Wang (2008) finding the bounds for currency cross-rate options, Chang et al. (2011) pricing and hedging quanto forward-starting floating-strike Asian options and Lee and Lee (2019) pricing symmetric type of power quanto options.

In addition, investors can exchange financial assets in the foreign markets at some time. To hedge the resulting risk, some kinds of financial tools should be developed. Among of these examples are quanto exchange options that help those domestic investors who do exchange financial assets in the foreign markets to hedge both the risk of exchanging financial assets and that of foreign exchange rate. Furthermore, investors can also exchange a financial asset in the foreign market for certain periods of time. In this case, the resulting risk of moving average price of a financial asset for certain periods of time can be hedged by quanto moving average exchange options, due to the fact is that: the moving average price of a financial asset for certain periods of time can be a representative price for that financial asset over exchanging periods of time. In this paper, we introduce one kind of these options, so called “Cross Currency Moving Average Exchange Options”, referred as CCMAE Options, and we also provide an analytic pricing formula for these newly innovated options. It allows an owner to exchange one domestic asset to another foreign asset at the average level of price for certain periods of time, mixing three-type options: quanto, moving average, and exchange options. Then, it shares all the benefits of them.

Quanto is an option whose payoff is related to one asset price multiplying a quantity, which is of course the foreign exchange rate here. This derivative has foreign underlying financial assets denominated in the domestic currency. This kind of option contains the forward foreign exchange rate to formulate the options which allow the investors to buy, for example, a risky financial asset in foreign country with the domestic currency. Derman, Karasinski and Wecker (1990) was the first paper to introduce the products and formed its pricing formulas. In their paragraph, they applied the partial differential equation to solve their pricing model, and found that the price of quanto option is related to the covariance between the price of foreign stock and the foreign exchange rate. Reiner (1992) introduced four types of quanto options. The differences of four options are mainly on the setting of the foreign exchange rate and the currency of strike price. The foreign exchange rate could be decided at maturity or in the beginning. The strike price can be denominated in the domestic or foreign currency. We obtain the idea from it and design our new contracts whose transferred foreign exchange rate is predetermined at a fixed level or floating with the foreign exchange market. On another similar product, Dravid, Matthew and Sun (1993) formed Nikkei Put Warrant pricing formula, the same formula as that of Derman, Karasinski and Wecker (1990) and Reiner (1992). Chen, Sears and Shahrokhi (1992) provided its pricing formula and offered some empirical evidence on it. Wei (1993) provided its pricing formula, and Wei (1994) had an empirical study of market efficiency on this quanto-type security.

An exchange option gives an owner the right to exchange one asset to another. It was first introduced by Margrabe (1978). Margrabe derived the pricing formula from Black-Scholes economy with a special skill named “numeraire” approach, as suggested by Steven Ross (see also 6th footnote in the original paper), using one of the asset prices as numeraire to convert the other asset price into the new price denominated in this numeraire. In this approach, he could formulate the option price for this exchange option within Black-Scholes framework.

Moving average options are complex path-dependent derivatives whose payoff depends on the moving average of stock prices (see also, Kao & Lyuu, 2003). Essentially, it is one type of Asian options, which settled by its average price of the underlying financial asset (see also, Ingersoll, 1987; Bouaziz et al., 1994;

Tsao, et al., 2003; Chung, et al., 2003; Chang & Tsao, 2011; Chang et al., 2011). Practically, most Asian derivatives on the markets are settled on the arithmetic average price. However, the geometric type Asian options are easier for pricing. In fact, there is no exact closed-form formula for pricing arithmetic average options. There are many different methods to price Asian options, including Monte Carlo simulations, analytic approximations, and PDE methods.

The well-known fact for arithmetic Asian pricing problem is essential that: the sum of lognormal distribution is no longer a lognormal distribution. To solve this problem, Turnbull and Wakeman (1991) chose Edgeworth expansion to approximate the real distribution of arithmetic average option. Levy (1992) applied Wilkinson approximation on the real distribution of such options, instead of Edgeworth expansion. The advantage is that this method is simpler than Turnbull and Wakeman approach and is just enough for approximating the value of arithmetic average option. Levy approximated the unknown arithmetic average distribution by the corresponding lognormal distribution, proved by the empirical evidence that the arithmetic average of lognormal distribution is close to the corresponding lognormal distribution. Milevsky and Posner (1998) proved that the sum of infinite lognormal random variables has the limiting distribution, called the reciprocal Gamma distribution. They could derive the closed-form solution from the reciprocal Gamma distribution to get the limiting price of arithmetic average option, and formulate the hedge ratio. Milevsky and Posner claimed that this method is faster and has the same precision.

Most importantly, Han et al. (2019) first introduced newly innovated financial products, to which it is essential to extend moving average options mentioned above -- moving average exchange options, and focused mainly on the method posted by Turnbull and Wakeman (1991). Moving average exchange options have share the same pricing problem of Asian options. They derived the exact closed-form formula for geometric moving average exchange options. However, the common options on the markets with the property of average payoff are almost arithmetic average ones. It is essential to have a pricing formula for the arithmetic type options. Fortunately, they also derive the analytic closed-form formula to approximate the value of the arithmetic moving average exchange options, and its accuracy has been proven by results of Monte Carlo simulation.

In this paper, we extend this seminal paper of Han et al. (2019) to examine two types of CCMAE options. One payoff is specified as fixed foreign exchange rate, which refers as a fixed foreign exchange rate CCMAE option (Fixed-CCMAE option); the other is specified as floating foreign exchange rate, which refers as a floating foreign exchange rate CCMAE option (Floating-CCMAE option). In other words, our main contribution is to extend moving average exchange options to quanto moving average exchange options that help those domestic investors who do exchange a financial asset for certain periods of time to hedge both the risk of exchanging financial assets and that of foreign exchange rate over the exchanging periods of time. We provide closed-form solutions for CCMAE options. One is an exact solution for each geometric type option and the other is an approximating solution for each arithmetic type option. It is ready to derive Greek letters for CCMAE options. It is essential important for practitioners to hedge the risk for issuing these options. Furthermore, we validate the accuracy of analytic formula of CCMAE options via Monte Carlo simulation, Monte Carlo integration, and numerical integration approaches.

The rest of the paper is organized as follows. Section 2 describes the general economic settings for the option pricing models. Section 3 derives the pricing formulas for CCMAE options. Section 4 provides numerical results which validate the accuracy of analytic formula of CCMAE options via Monte Carlo simulation, Monte Carlo integration, and numerical integration approaches. Section 5 concludes.

THE ECONOMIC SETTING

We consider a simplified continuous economy which is combined both domestic and foreign markets. We assume both the domestic and foreign risk-free rates are non-negative constants over the investment period. There should be a foreign saving account which means the value of investing one dollar in foreign currency to grow in a foreign risk-free rate over time. Then, the value of foreign saving account B_t^* at time t could be expressed as:

$$dB_t^* = r^* B_t^* dt, \tag{1}$$

where r^* means the instantaneous foreign risk-free rate. We also define the domestic saving account：

$$dB_t = r B_t dt , \tag{2}$$

where r means the instantaneous domestic risk-free rate.

In addition, we could simplify the stock markets as only two risky assets in our world: a domestic and a foreign stock. Each one is denominated in the currency of their issuing country. That is: the foreign stock price S^* is denominated in the foreign currency, and the domestic stock price S is denominated in the domestic currency. We assume both stock price dynamics follow geometric Brownian motions:

$$dS_t^* = \mu_{S^*} S_t^* dt + \sigma_{S^*} S_t^* dW_{2t}^P , \tag{3}$$

$$dS_t = \mu_S S_t dt + \sigma_S S_t dW_{3t}^P , \tag{4}$$

where $$dW_{2t}^P dW_{3t}^P = \rho_{23} dt . \tag{5}$$

Here, each one of dW_{2t}^P and dW_{3t}^P means the unpredictable part of the foreign and domestic stocks, respectively. They are standard Brownian motions with correlation ρ_{23} .

The foreign exchange rate process x_t is a conversion ratio of one dollar of foreign currency to domestic currency with the assumption of the following equations (see also, Garman and Kohlhagen, 1983):

$$dx_t = \mu_x x_t dt + \sigma_x x_t dW_{1t}^P , \tag{6}$$

where $$dW_{1t}^P dW_{2t}^P = \rho_{12} dt , \tag{7}$$

$$dW_{1t}^P dW_{3t}^P = \rho_{13} dt . \tag{8}$$

Here, we emphasis that all the innovation terms (e.g. dW_{it}^P , $i = 1, 2, 3$) in these stochastic differential equations are under the physical probability measure.

The domestic price of foreign stock will be the product of the foreign stock price and the foreign exchange rate:

$$u_t = x_t S_t^* . \tag{9}$$

We can obtain that:

$$du_t = r\,u_t dt + \sigma_x u_t dW_{1t}^Q + \sigma_{S^*} u_t dW_{2t}^Q . \tag{10}$$

The domestic price of foreign saving account is the product of the foreign saving account value and the foreign exchange rate:

$$m_t = x_t B_t^* . \tag{11}$$

We can also see that:

$$dm_t = r\,m_t dt + \sigma_x m_t dW_{1t}^Q . \tag{12}$$

Note that: the economic environment is assumed to be settings of the ideal market:

(a) The short rates are constant,
(b) No dividends,
(c) Frictionless markets (no tax, no transaction cost, etc.),
(d) No restrictions on short sells.

Also, markets are assumed to be efficient and without arbitrage opportunities.

Let P_t denote the spot price of underlying asset at time t. For the n observations of underlying price, then the geometric and arithmetic average price could be defined as follows, respectively:

Arithmetic average price:

$$A_n(P,T) = \frac{1}{n}\sum_{i=1}^{n} P_{T-n+i} . \tag{13}$$

Geometric average price:

$$G_n(P,T) = \left(\prod_{i=1}^{n} P_{T-n+i}\right)^{\frac{1}{n}} . \tag{14}$$

where P_t denotes the stock price, e.g. S_t^*, S_t, or u_t.

In the contract, n is specified by the length of moving window for a foreign stock and m is specified as the length of moving window for a domestic stock. Then, the payoff of each CCMAE option could be defined as one of the following four types:

(a) Arithmetic average type of Floating - CCMAE options:

$$FAC_T = \left[A_n\left(xS^*,T\right) - A_m(S,T)\right]^+ . \tag{15}$$

(b) Geometric average type of Floating - CCMAE options:

$$FGC_T = \left[G_n\left(xS^*, T\right) - G_m(S,T) \right]^+ . \quad (16)$$

(c) Arithmetic average type of Fixed - CCMAE options:

$$GAC_T = \left[A_n\left(\overline{x}S^*, T\right) - A_m(S,T) \right]^+ . \quad (17)$$

(d) Geometric average type of Fixed - CCMAE options:

$$GGC_T = \left[G_n\left(\overline{x}S^*, T\right) - G_m(S,T) \right]^+ . \quad (18)$$

Also, the foreign exchange rate and the foreign stock price dynamics in domestic risk-neutral probability measure are (see also, Dravid, Matthew & Sun, 1993):

$$dx_t = \left(r - r^*\right) x_t dt + \sigma_x x_t dW_{1t}^Q , \quad (19)$$

$$dS_t^* = \left(r^* - \rho_{12}\sigma_x\sigma_{S^*}\right) S_t^* dt + \sigma_{S^*} S_t^* dW_{2t}^Q , \quad (20)$$

where

$$dW_{1t}^Q dW_{2t}^Q = \rho_{12} dt , \quad (21)$$

$$dW_{1t}^Q = \left(\frac{\mu_x + r^* - r}{\sigma_x} \right) dt + dW_{1t}^P , \quad (22)$$

$$dW_{2t}^Q = \left(\frac{\mu_{S^*} - r^* + \rho_{12}\sigma_x\sigma_{S^*}}{\sigma_{S^*}} \right) dt + dW_{2t}^P . \quad (23)$$

OPTIONS PRICING FORMULA

1. Floating Foreign Exchange Rate Options

Consider each floating foreign exchange rate option as the moving average exchange option with two domestic assets. And, let

$$u_t = x_t S_t^* , \quad (24)$$

where u_t is the foreign stock price denominated in the domestic currency.

From Ito's lemma:

$$\frac{du_t}{u_t} = r\,dt + \sigma_x dW_{1t}^Q + \sigma_{S^*} dW_{2t}^Q . \tag{25}$$

Then, the quadratic variation of this random part would be

$$\left[\sigma_x dW_{1t}^Q + \sigma_{S^*} dW_{2t}^Q, \sigma_x dW_{1t}^Q + \sigma_{S^*} dW_{2t}^Q\right] = \sigma_x^2 dt + \sigma_{S^*}^2 dt + 2\rho_{12}\sigma_x\sigma_{S^*} dt . \tag{26}$$

Hence,

$$\frac{du_t}{u_t} = r\,dt + \sqrt{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}\; dW_{ut}^Q , \tag{27}$$

where W_{ut}^Q is a standard Brownian motion under the risk-neutral probability measure, and its variance rate is denoted by σ_u^2.

Again, the dynamic of domestic stock price is:

$$\frac{dS}{S} = r\,dt + \sigma_S dW_{3t}^Q . \tag{28}$$

Hence, one could obtain the correlation between the domestic price of foreign stock and the domestic stock price.

$$dW_{3t}^Q dW_{ut}^Q = \frac{\rho_{13}\sigma_x + \rho_{23}\sigma_{S^*}}{\sqrt{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}} dt \tag{29}$$

That is, the correlation between dW_{3t}^Q and dW_{ut}^Q is denoted by ρ_{u3}.

$$\rho_{u3} = \frac{\rho_{13}\sigma_x + \rho_{23}\sigma_{S^*}}{\sqrt{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}} \tag{30}$$

In the pricing theory, an option written on geometric average is far more easily to have a pricing formula than arithmetic average one. For the reason is that the stock price at any time would be a lognormal distribution with mean and variance related to the time passed through. After taking the logarithm, the result of the logarithm of geometric average stock price would be the sum of normal random variables, which is still a normal distribution. However, only approximation could be obtained for the value of arithmetic average option in the financial literature. In this paper, we first work on an exact solution for each geometric average option, and then go to an approximate solution for each arithmetic average option.

Turnbull and Wakeman developed a formula for geometric average price distribution. In their framework, we could obtain:

$$\ln G_n(u) : N\left(\mu_{G(u)}, \sigma^2_{G(u)}\right), \tag{31}$$

where

$$\begin{aligned}\mu_{G(u)} = \ln u_0 &+ \left(r - \frac{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}{2}\right)(T-n) \\ &+ \left(r - \frac{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}{2}\right)\sum_{j=1}^{n}\frac{j}{n}\end{aligned}, \tag{32}$$

$$\sigma^2_{G(u)} = \left(\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}\right)\left[(T-n) + \sum_{j=1}^{n}\frac{j^2}{n^2}\right], \tag{33}$$

$$\ln G_m(S) : N\left(\mu_{G(S)}, \sigma^2_{G(S)}\right) \tag{34}$$

where

$$\mu_{G(S)} = \ln S_0 + \left(r - \frac{\sigma_S^2}{2}\right)(T - m) + \left(r - \frac{\sigma_S^2}{2}\right)\sum_{j=1}^{m}\frac{j}{m}, \tag{35}$$

$$\sigma^2_{G(S)} = \sigma_S^2\left[(T - m) + \sum_{j=1}^{m}\frac{j^2}{m^2}\right]. \tag{36}$$

There are three cases in the relationship between n and m. These just give the flexibility for investors and issuers to determine the variant contracts. In fact, market participants prefer longer moving average window of CCMAE option on the following cases: larger volatility of stock price, less liquidity in the markets, more vulnerable to market manipulation. The covariances between these two distributions are determined by the relationship of observation periods of time, say n and m:[1]

Case 1: the same observations (usual case) ($n = m$)

$$\begin{aligned}&\text{cov}\left(\ln G_n(u), \ln G_m(S)\right) \\ &= \sigma_u\sigma_S\rho_{u3}\left[(T - n) + \frac{1}{6n}(n+1)(2n+1)\right]\end{aligned} \tag{37}$$

Case 2: the observations of foreign stock are more than those of domestic stock

1 For more details, please refer to Appendix.

$(n > m)$

$$\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right) = \sigma_u \sigma_S \rho_{u3} \times \left[(T - n) + \frac{1}{2n}(n + m + 1)(n - m) + \frac{1}{6n}(m + 1)(2m + 1) \right] \tag{38}$$

Case 3: the observations of foreign stock are less than those of domestic stock $(n < m)$

$$\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right) = \sigma_u \sigma_S \rho_{u3} \times \left[(T - m) + \frac{1}{2m}(n + m + 1)(m - n) + \frac{1}{6m}(n + 1)(2n + 1) \right] \tag{39}$$

The correlation of bivariate normal distribution could be obtained by:

$$\rho_{GG} = \frac{\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right)}{\sigma_{G(u)}\sigma_{G(S)}}. \tag{40}$$

Hence, we obtain the density function of geometric average price:

$$\begin{aligned} f\left(G_n(u), G_m(S)\right) &= \frac{1}{2\pi G_n(u) G_m(S) \sigma_{G(u)} \sigma_{G(S)} \left(1 - \rho_{GG}^2\right)} \\ &\times \exp\left\{ -\frac{1}{2\left(1 - \rho_{GG}^2\right)} \right. \\ &\times \left[\left(\frac{\ln G_n(u) - \mu_{G(u)}}{\sigma_{G(u)}} \right)^2 + \left(\frac{\ln G_m(S) - \mu_{G(S)}}{\sigma_{G(S)}} \right)^2 \right. \\ &\left.\left. - 2\rho_{GG} \left(\frac{\ln G_n(u) - \mu_{G(u)}}{\sigma_{G(u)}} \right) \left(\frac{\ln G_m(S) - \mu_{G(S)}}{\sigma_{G(S)}} \right) \right] \right\}. \end{aligned} \tag{41}$$

After solving the distribution of geometric average price, by the classic framework of pricing Asian options, we could discount the expectation of the option payoff under risk-neutral probability measure to get the current option price for each of n and m, respectively:

$$FGC_0 = e^{-rT} E_0^Q\left[\left(G_n(u) - G_m(S)\right)^+\right]. \tag{42}$$

It seems difficult to extract more explicit form of the expectation or integration. One could apply numerical integration later to evaluate the option value with the specific parameters by comparing with the other pricing methods proposed in this paper.

2. Fixed Foreign Exchange Rate options

With a fixed foreign exchange rate, the dynamics would be multiplied by a constant $\bar{x}$.

$$d\left(\bar{x}\, S_t^*\right) = \bar{x} S_t^*\left[\left(r^* - \rho_{12}\sigma_x\sigma_{S^*}\right)dt + \sigma_{S^*} dW_{2t}^Q\right] \tag{43}$$

Let's set the fixed rate foreign stock price as

$$v_t = \bar{x}\, S_t^*, \tag{44}$$

$$dv_t = \left(r^* - \rho_{12}\sigma_x\sigma_{S^*}\right)v_t dt + \sigma_{S^*} v_t dW_{2t}^Q. \tag{45}$$

The dynamic of v substitutes for the dynamic of u when we discuss the fixed foreign exchange rate options. By the same pricing procedure as floating foreign exchange rate options, we could form the mean and the variance of the logarithm of geometric average price. Then, we have the covariance depending on the observation periods of time, say n and m. We also obtain the option price by discounting the expected payoff, which can be solved by numerical integration of the density function of geometric average price.

3. Closed-form Analytic Solutions

Although the geometric average price is distributed from a lognormal distribution, the dynamics of geometric average price are also geometric Brownian motions.

Recall the original exchange option pricing formula in Margrabe (1978):

$$w(x_1, x_2, t) = x_1 N(d_1) - x_2 N(d_2), \tag{1}$$

where

$$d_1 = \frac{\ln\left(\frac{x_1}{x_2}\right) + \frac{1}{2}\sigma^2\left(t^* - t\right)}{\sigma\sqrt{t^* - t}}, \tag{2}$$

$$d_2 = \frac{\ln\left(\frac{x_1}{x_2}\right) - \frac{1}{2}\sigma^2\left(t^* - t\right)}{\sigma\sqrt{t^* - t}} = d_1 - \sigma\sqrt{t^* - t}, \tag{3}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2. \tag{4}$$

3.1 Floating rate geometric average exchange options

$$FGC_0 = E_0^Q\left[\frac{G_n(u)}{B_T}\right]N(d_1) - E_0^Q\left[\frac{G_m(S)}{B_T}\right]N(d_2), \tag{5}$$

where

$$d_1 = \frac{\ln\left\{\frac{E\left[G_n(u)\right]}{E\left[G_m(S)\right]}\right\} + \frac{\sigma_P^2}{2}}{\sigma_P}, \tag{6}$$

$$d_2 = d_1 - \sigma_P, \tag{7}$$

$$\sigma_P^2 = \sigma_{G(u)}^2 + \sigma_{G(S)}^2 - 2\rho_{GG}\sigma_{G(u)}\sigma_{G(S)}. \tag{8}$$

3.2 Fixed rate geometric average exchange options

$$GGC_0 = E_0^Q\left[\frac{G_n(v)}{B_T}\right]N(d_1) - E_0^Q\left[\frac{G_m(S)}{B_T}\right]N(d_2), \tag{9}$$

where

$$d_1 = \frac{\ln\left\{\frac{E\left[G_n(v)\right]}{E\left[G_m(S)\right]}\right\} + \frac{\sigma_P^2}{2}}{\sigma_P}, \tag{10}$$

$$d_2 = d_1 - \sigma_P, \tag{11}$$

$$\sigma_P^2 = \sigma_{G(v)}^2 + \sigma_{G(S)}^2 - 2\rho_{GG}\sigma_{G(v)}\sigma_{G(S)}. \tag{12}$$

which means to shoot the target distribution at maturity from the discounted of the expectation of the moving average price at maturity. Since the non-Markovian property of moving average price, if we replace the current stock price in the formula of Margrabe (1978) with the current geometric moving average price, the pricing formula should depend not only the current stock price but also historical stock prices. Instead we replace with the current value of that of the expected geometric moving average at maturity under the risk-neutral distribution on hand and then discounting it to the current time. With applying Margrabe formula, we know that the non-Markovian property would have no influence on the accuracy of the pricing model (see also, Han et al. 2019).

3.3 Floating rate arithmetic average exchange options

We know from the financial literature that an arithmetic average price is close to its corresponding geometric average price. So, we could choose some of the parameters of geometric average price distribution to approximate the unknown arithmetic average price distribution.

Here, we apply Wilkinson approximation of Levy (1992) on these pricing models. However, the well-known fact is that the sum of lognormal distribution is no longer a lognormal distribution. Levy proved by the empirical evidence that the arithmetic average distribution is close to its corresponding lognormal distribution. Hence, Levy approximated the unknown arithmetic average distribution by its corresponding lognormal distribution. We assume that the logarithm of arithmetic average price $\ln A(t)$ is from a normal distribution with the mean μ_A and the variance σ_A^2, then

$$E\left[e^{\lambda \ln A(t)}\right] = E\left[A(t)^{\lambda}\right] = e^{\lambda \mu_A + \frac{1}{2}\lambda^2\sigma_A^2}. \tag{13}$$

When $\lambda = 1, 2$, we could have the unknown parameters, the mean μ_A and the variance σ_A^2, on the approximating lognormal distribution:

$$\mu_A = 2\ln E\left[A(t)\right] - \frac{1}{2}\ln E\left[A(t)^2\right], \tag{14}$$

$$\sigma_A^2 = \ln E\left[A(t)^2\right] - 2\ln E\left[A(t)\right]. \tag{15}$$

By applying from the equations (13) -- (15), we would have the mean and the variance of the arithmetic average price distribution with the simple recursive formula of Turnbull and Wakeman (1991), and then we use the correlation of geometric average price distribution as an approximation to that of arithmetic average price distribution. After that, we just go back to the same pricing procedure as previous sections. This pricing procedure is shown good enough based on comparing pricing results of Monte Carlo simulation (see also, Han et al. 2019).

NUMERICAL RESULTS

Monte Carlo simulation is one of the most wildly accepted methods in pricing complicated exotic options. After insuring that our stock price and foreign exchange rate dynamics are reasonable chosen and the derivatives we have chosen is martingale under the proper numeraire and its corresponding measure, we could have the confidence that the method of Monte Carlo simulation could produce an approximate solution for the complicated exotic option. And, in theory, this solution would approach to the real solution for that option with an infinitesimal sampling. Hence, we choose the method of Monte Carlo simulation as our starting point for pricing our options. However, the shortages of Monte Carlo simulation, such as time consuming and having the random property on its result, made us have the necessary task to seek for other effective pricing methods.

In this paper, we then apply techniques of numerical integration to obtain the pricing results. There are several numerical integration approaches to do the integration for options, actually. Here, we choose iterated integration, and the other of Monte Carlo integration, as our pricing tools to solve the integration for options. The iterated integration has a great advantage that the result of this integration has no sampling error and is a certain value, and hence a non-random term inherently. Even more, we could control the pricing error by a very small size of width in the numerical integration. This precision makes it as another benchmark for comparing pricing methods. However, Monte Carlo integration is a relatively simpler way of numerical integration. Although, this method could generate many random solutions, we could use these solutions to check if the solution for our options is accurate.

To this end, we also provide the pricing results of our analytic formula for options. To implement these methods, we list all the base-line parameters for pricing models on Table 1 and 2, and we would apply our pricing models with these parameters if not mentioned.

In the following, we compare results on the certainty methods of iterated integration and analytic formula and those on the uncertainty methods of Monte Carlo simulation and Monte Carlo integration, as shown in Table 3 and 4. We found that all solutions of these certainty methods are the same among all different parameter settings. This provides strong evidence that the analytic formula is perfect in normal case. It is worth to mention that all the pricing results are very close. Among all the methods we mentioned above, the analytic formula is the fastest way to obtain the option price[2]. We also see that the foreign stock price would have a positive effect on our options while the domestic stock price has a negative effect.

However, no significant evidence shows that any one of the pricing methods was exactly higher or lower than another one. With the nature of Monte Carlo simulation and Monte Carlo integration methods, they both have random results on option price. We plot the results of Monte Carlo simulation by dots, and draw the line for those results of analytic formula for our options. We could see that the curve of analytic solutions passes through those results of Monte Carlo simulation as shown in Figure 1. In other words, this figure justifies the reliable and correctness of our analytic formula for our options.

The fixed foreign exchange rate options are always a step lower than the floating foreign exchange rate options. In our study, this propety is although not always the case and depends on the background of parameter settings. But, in our economics, the foreign risk-free rate is 10% while the domestic saving account

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[2] We provide an accurate closed-form solution, just as exchange formula, and further an analytic approximate closed-form solution which requires some iterations of calculation (see Han et al. 2019). Hence, the computation time of these analytic formula is far less than that of another pricing method mentioned above, which requires simulation to work hard and consumes a lot of time.

Table 1 The default parameters of our model

Variables	Descriptions	Default values
S^*	Foreign stock (in foreign currency)	50
S	Domestic stock	50
x	Foreign exchange rate	1
$\bar{x}$	Fixed foreign exchange rate	1
r^*	Foreign risk-free rate	0.1
r	Domestic risk-free rate	0.05
σ_x	Foreign exchange rate volatility	0.05
σ_{S^*}	Foreign stock volatility	0.2
σ_S	Domestic stock volatility	0.2
ρ_{12}	Correlation between the foreign stock price and the foreign exchange rate	0.05
ρ_{13}	Correlation between the domestic stock price and the foreign exchange rate	-0.05
ρ_{23}	Correlation between the foreign stock price and the domestic stock price	0.05

(a) The stock price and the foreign exchange rate listed here are the initial value at current time. They would fluctuate with time.
(b) The correlations between the stock price and the foreign exchange rate are set to comply with the common assumption that the currency would be raised when stock market in their country would rise the stock price.
(c) The foreign exchange rate is simply set to be one to avoid complication, hence the initial price of foreign stock could be considered as both in the foreign price and in the domestic price, initially.

Table 2 Other parameters

Variables	Descriptions	Default values
T	Time to maturity (years)	0.5
paths	The number of paths for Monte Carlo simulation	10000
n	Foreign stock observation periods of time	30
m	Domestic stock observation periods of time	30
repeats	The number of iterations for Monte Carlo simulation and integration	30
cases	The number of sampling cases for Monte Carlo integration	50000

(a) We assume that there are 250 trading days in a year; hence for half year we have 125 simulation steps to simulate the stock price fluctuation.
(b) Considering the computation time, we could easily have 10000 simulation paths to obtain the expectation within few seconds, which is quick enough. However, its precision may not be good enough, comparing to our analytic formula. If needed, we would do more paths to have a more precise answer.

Table 3 Floating foreign exchange rate options
(with change in initial foreign stock price)

Foreign Stock Price	44	46	48	50	52	54	56	58	60
Monte Carlo Simulation	1.231	1.876	2.665	3.634	4.806	6.098	7.594	9.173	10.825
Standard deviation	0.037	0.031	0.039	0.043	0.052	0.058	0.077	0.088	0.081
Iterated integration	1.241	1.871	2.676	3.658	4.812	6.124	7.575	9.147	10.819
Monte Carlo integration	1.240	1.869	2.671	3.662	4.815	6.126	7.575	9.148	10.826
Standard deviation	0.015	0.019	0.016	0.026	0.024	0.025	0.038	0.034	0.039
Analytic formula	1.241	1.871	2.676	3.658	4.812	6.124	7.575	9.147	10.819

The other parameters except initial foreign stock price are the base-line parameters on Table 1 and 2.

Table 4 Floating foreign exchange rate options
(with change in initial domestic stock price)

Domestic Stock Price	44	46	48	50	52	54	56	58	60
Monte Carlo simulation	16.111	14.297	12.500	10.823	9.293	7.817	6.527	5.378	4.393
Standard deviation	0.071	0.100	0.068	0.063	0.081	0.082	0.074	0.062	0.086
Iterated integration	16.125	14.274	12.498	10.819	9.254	7.821	6.530	5.386	4.390
Monte Carlo integration	16.135	14.288	12.496	10.820	9.256	7.824	6.531	5.376	4.394
Standard deviation	0.034	0.051	0.031	0.032	0.044	0.039	0.036	0.038	0.022
Analytic formula	16.125	14.274	12.498	10.819	9.254	7.821	6.530	5.386	4.390

The other parameters except initial domestic stock price are the base-line parameters on Table 1 and 2.

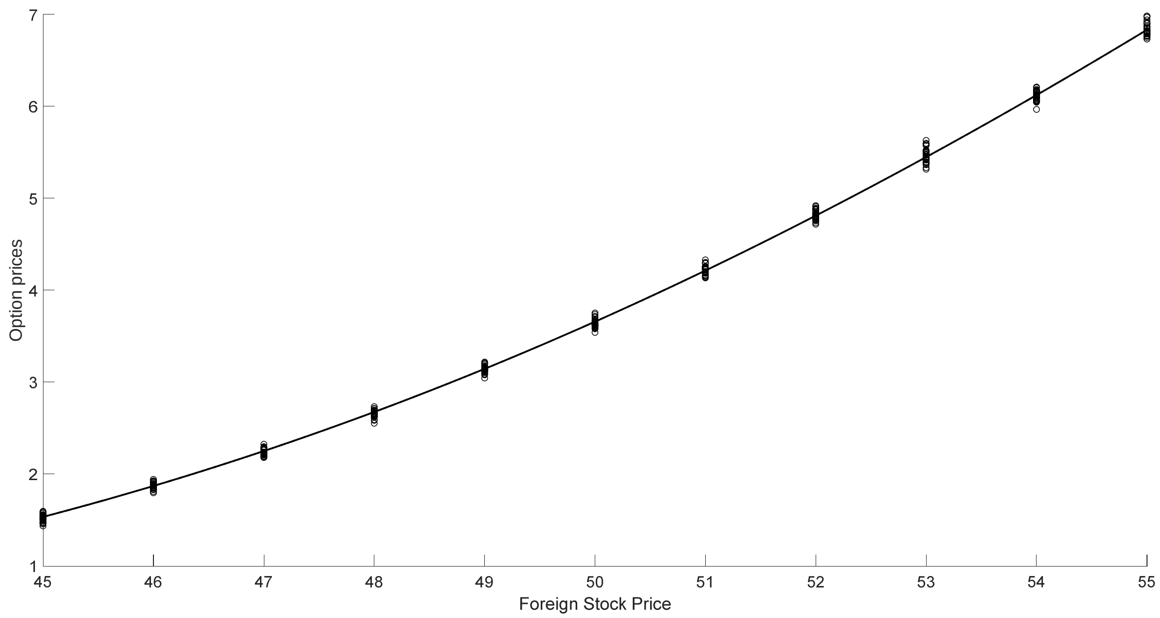

Fig. 1. Monte Carlo simulation by dots, and draw the line for analytic formula.

only earns 5% annualy. The interest rate parity tells us that the foreign exchange rate should fall since the spread $(r - r^*)$ is negative if no arbitrage opportunity holds in the markets.

If we apply the classic martingale option pricing approach, the method will imply that we should accept the interest rate parity assumption (IRP assumption) in our models. Consider the trend of the foreign exchange rate, the foreign stock price would fall more than in its own country when converting to the domestic currency. However, in the martingale approach, it would adjust the yield of the foreign stock in the domestic currency back to risk-free rate under risk-neutral probabilty measure. Hence, the fixed foreign exchange rate option price would be conversly raised by the measure change. This may explain one of the most important difference between two contracts.

The comparison results in Table 5 tell us that the difference of floating foreign exchange rate and fixed foreign exchange rate is not actually large. It means that the premium of the fixed foreign exchange rate only has a relatively small portion in option value. It may be caused by our settings of relatively small volatility on the foreign exchange rate, which is only one fourth of the stock price volatility.

As we could see in Table 6 and 7, the foreign risk-free interest rate has no effect on the floating foreign exchange rate options. This characteristic does not comply to intuition, but it’s not so surprising under the risk-neutral world. We could consider the foreign stock price times the foreign exchange rate as a pure domestic financial asset. Hence, it should have grown with the domestic risk-free interest rate under the risk-neutral world. The foreign risk-free rate would be adjusted by the dynamic of foreign exchange rate which follows IRP assumption. To this end, we could really expect that the foreign risk-free rate does not represent in the pricing formula of the floating foreign exchange rate options. However, the foreign risk-free rate does affect the fixed foreign exchange rate options. Since the dynamic of foreign stock price in the fixed foreign exchange rate options contains the drift term $(r^* - \rho_{12}\sigma_x\sigma_{S^*})$, the foreign risk-free interest rate would directly change the expectation of the foreign stock price at maturity, hence it’s obviously that the fixed exchange rate options would be affected by the foreign risk-free rate.

Table 5 Floating vs Fixed foreign exchange rate options

Foreign Stock Price	30	40	50	60	70	80	90	100
floating rate	0.006	0.449	3.658	10.819	20.077	29.918	39.867	49.832
fixed rate	0.004	0.405	3.550	10.741	20.042	29.899	39.850	49.813

The other parameters except initial foreign stock price are the base-line parameters on Table 1 and 2.

Table 6 Interest rate sensitivity analysis for Floating CCMAE options

r^* \ r	0%	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
0 %	3.669	3.667	3.665	3.663	3.660	3.658	3.656	3.654	3.652	3.650	3.648
5 %	3.669	3.667	3.665	3.663	3.660	3.658	3.656	3.654	3.652	3.650	3.648
10 %	3.669	3.667	3.665	3.663	3.660	3.658	3.656	3.654	3.652	3.650	3.648

Table 7 Interest rate sensitivity analysis for Fixed CCMAE options

r^* \ r	0%	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%
0 %	3.560	3.441	3.325	3.212	3.102	2.994	2.889	2.786	2.687	2.589	2.495
5 %	4.185	4.052	3.922	3.795	3.671	3.550	3.431	3.316	3.203	3.093	2.985
10 %	4.879	4.732	4.588	4.447	4.308	4.173	4.041	3.911	3.784	3.661	3.540

The domestic interest rate would have some effects on the floating foreign exchange rate options; however, the effect is small comparing to the fixed foreign exchange rate options. Due to the fact is that the foreign stock price would not grow with the domestic risk-free rate, instead it grows with the foreign risk-free interest rate minus the covariance between the foreign stock price and the foreign exchange rate.

In Table 8, the implication of correlation has a little complicated effect on the option price. Due to 1990s, traders on quanto options still confused why the covariance between the foreign stock price and the foreign exchange rate would affect the option price while this kind of options have nothing to do with the foreign exchange rate after purchasing, and that’s why the buyer buys these options because they don’t like the foreign exchange rate risk. There are a lot of financial literature on this confusing issue including Derman, Karasinski and Wecker (1990) and Dravid, Matthew and Sun (1993). We post the results and discuss this issue to complete our analysis.

Table 8 Correlation sensitivity for CCMAE options

Foreign stock and exchange rate	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**
Floating rate	3.354	3.451	3.545	3.636	3.725	3.812	3.897	3.980
Fixed rate	3.627	3.603	3.579	3.556	3.532	3.509	3.486	3.462
Between stocks	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**
Floating rate	4.689	4.398	4.086	3.748	3.375	2.956	2.465	1.848
Fixed rate	4.604	4.308	3.989	3.642	3.258	2.821	2.303	1.628
Domestic stock and exchange rate	**-0.6**	**-0.4**	**-0.2**	**0**	**0.2**	**0.4**	**0.6**	**0.8**
Floating rate	3.900	3.814	3.726	3.636	3.543	3.448	3.351	3.250
Fixed rate	3.550	3.550	3.550	3.550	3.550	3.550	3.550	3.550

The most important correlation is the correlation between the foreign stock price and the foreign exchange rate. One may consider this correlation as “the correlation risk”. Since this parameter of the joint distribution of the foreign stock price and the foreign exchange rate do affect the option price sometimes. As we could see that the correlation would have reverse effect on different contracts: positive on the floating foreign exchange rate options and negative on the fixed foreign exchange rate options. It’s very interesting to see that they are intersect about on the correlation -0.2 when we fixed all the other parameters. Essentially, the value of floating foreign exchange rate options is higher than that of fixed foreign exchange rate options when the coefficient of correlation is positive. That is: when the correlation is big enough, it would reduce the yield of foreign stock price in the fixed foreign exchange rate options and add up the volatility in the floating foreign exchange rate options.

For the correlation between two stock prices, it’s relatively simple. Since we can expect the volatility will be smaller when the correlation between two stock prices is higher. These effects on two contracts are merely the same.

For the correlation between the domestic stock price and the foreign exchange rate, the correlation could have reverse effect on different contracts: negative on the floating foreign exchange rate options and no influence on the fixed foreign exchange rate options. On equation (25) and (28), we can see the floating foreign exchange rate options have negative effect simply because highly correlation causes the co-movement between the domestic currency denominated foreign stock and domestic stock dynamics. Hence, higher correlation between the domestic stock price and the foreign exchange rate is, less value the option is. For the same reason, while the correlation between the domestic stock price and the foreign exchange rate does not appear in equation (43), the fixed foreign exchange rate options have no effect about the correlation change between the domestic stock price and the foreign exchange rate.

CONCLUSION

In this paper, our main contribution is to extend moving average exchange options to quanto moving average exchange options that help those domestic investors who do exchange a financial asset for certain periods of time to hedge both the risk of exchanging financial assets and that of foreign exchange rate over the exchanging periods of time. Four possible new types of custom-tailored exotic options are provided to complete the financial markets, which are: (a) Floating Cross Currency Moving Average Exchange options; (b) Fixed Cross Currency Moving Average Exchange options, with the different types of average.

The closed-form formulas for CCMAE options are provided. One is an exact solution for each geometric type option and the other one is an approximate solution for each arithmetic type option. We justify the reliable and correctness of our analytic formula by Monte Carlo simulation, Monte Carlo integration, and iterated integration approaches.

Numerical analysis shows that the foreign stock price and the domestic stock price have different effects on the option price. Also, the correlations between three dynamics have different effects on the option price. The most important correlation discussed here is the correlation between the foreign stock price and the foreign exchange rate. This correlation is positive on the floating foreign exchange rate options but negative on the fixed foreign exchange rate options. The correlation between two stock prices has a negative effect on both option values, and the correlation of the domestic stock price and the foreign exchange rate has negative effect on the floating foreign exchange rate options but don’t have any effect on the fixed foreign exchange rate options.

The applications of CCMAE options may include: (a) a spread trade between the stock and its corresponding depositary receipt, and (b) financial tools for domestic investors who invest in the foreign countries. The risk of spread trade is hedged by the moving average exchange option, and the different risk of investing in the foreign stock or asset is then hedged by the quanto type of these options. It is hoped that these options we provide here offer new applications for the financial markets and encourage others for the future research.

APPENDIX

We assume both stock price dynamics follow geometric Brownian motions:

$$dS_t^* = \mu_{S^*} S_t^* dt + \sigma_{S^*} S_t^* dW_{2t}^P, \tag{A.1}$$

$$dS = \mu_S S dt + \sigma_S S dW_{3t}^P, \tag{A.2}$$

$$dW_{2t}^P dW_{3t}^P = \rho_{23} dt. \tag{A.3}$$

The foreign exchange rate process x_t is a conversion ratio of one dollar of foreign currency to domestic currency with the assumption of the following equations (see also, Garman and Kohlhagen, 1983):

$$dx_t = \mu_x x_t dt + \sigma_x x_t dW_{1t}^P, \tag{A.4}$$

$$dW_{1t}^P dW_{2t}^P = \rho_{12} dt, \tag{A.5}$$

$$dW_{1t}^P dW_{3t}^P = \rho_{13} dt. \tag{A.6}$$

The domestic price of foreign stock will be the product of the foreign stock price and the foreign exchange rate:

$$u_t = x_t S_t^*, \tag{A.7}$$

$$du_t = r\, u_t dt + \sigma_x u_t dW_{1t}^Q + \sigma_{S^*} u_t dW_{2t}^Q. \tag{A.8}$$

In the following paragraph, we derive the key parameters which are the covariances of returns between two geometric average prices. One is the foreign stock geometric average price denominated in domestic currency with floating foreign exchange rate, and the other one is domestic stock geometric average price. We already know both price dynamics follow geometric Brownian motions:

$$dS_t = r\, S_t dt + \sigma_S S_t dW_{3t}^Q, \tag{A.9}$$

$$\begin{aligned} du_t &= r\, u_t dt + \sigma_u u_t dW_{u\,t}^Q, \\ &= r\, u_t dt + \left(\sigma_x u_t dW_{1t}^Q + \sigma_{S^*} u_t dW_{2t}^Q\right) \end{aligned} \tag{A.10}$$

The stochastic differential equation could be rewritten under the following discrete-type model[3]:

$$S_t = S_{t-1} \exp\left[\left(r - \frac{\sigma_S^2}{2}\right) + \sigma_S \Delta W_{3t}^Q\right], \tag{A.11}$$

$$u_t = u_{t-1} \exp\left[\left(r - \frac{\sigma_u^2}{2}\right) + \sigma_u \Delta W_{ut}^Q\right] \tag{A.12}$$

where $\Delta W_{it}^Q \sim N(0,1)$, $i = 3, u$.

Since the geometric average prices are log-normally distributed, we could easily derive the distributions of geometric average prices by applying the method of Turnbull and Wakeman (1991):

$$\begin{aligned} \ln G_n(u) &= \ln u_0 \\ &+ \left(r - \frac{\sigma_u^2}{2}\right)\left[(T - n) + \frac{1}{n}\sum_{t=T-n+1}^{T}(T + 1 - t)\right] \\ &+ \sigma_u\left[\sum_{t=1}^{T-n}\Delta W_{ut}^Q + \frac{1}{n}\sum_{t=1}^{n}(n - t + 1)\Delta W_{u(T-n+t)}^Q\right]. \end{aligned} \tag{A.13}$$

The geometric average domestic price of foreign stock follows the lognormal distribution:

$$\ln G_n(u) \sim N\left(\mu_{G(u)}, \sigma_{G(u)}^2\right), \tag{A.14}$$

where

$$\begin{aligned} \mu_{G(u)} &= \ln u_0 + \left(r - \frac{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}{2}\right)(T - n) \\ &+ \left(r - \frac{\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}}{2}\right)\sum_{j=1}^{n}\frac{j}{n} \end{aligned}, \tag{A.15}$$

$$\sigma_{G(u)}^2 = \left(\sigma_x^2 + \sigma_{S^*}^2 + 2\rho_{12}\sigma_x\sigma_{S^*}\right)\left[(T - n) + \sum_{j=1}^{n}\frac{j^2}{n^2}\right]. \tag{A.16}$$

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3 For the discrete-type model, we rewrite the parameters as one period notation for convenience. That is we rewrite $r\,\Delta t$ as r and $\sigma^2\Delta t$ as σ^2 for one period notation.

It could parallel apply the same methods to the geometric average price of domestic stock:

$$\ln G_m(S) = \ln S_0 + \left(r - \frac{\sigma_S^2}{2}\right)\left[(T - m) + \frac{1}{m}\sum_{t=T-m+1}^{T}(T + 1 - t)\right] + \sigma_S\left[\sum_{t=1}^{T-m}\Delta W_{3t}^Q + \frac{1}{m}\sum_{t=1}^{m}(m - t + 1)\Delta W_{3(T-m+t)}^Q\right].$$

Then the distribution of geometric average price of domestic stock would become:

$$\ln G_m(S) \sim N\left(\mu_{G(S)}, \sigma_{G(S)}^2\right) \tag{A.17}$$

where

$$\mu_{G(S)} = \ln S_0 + \left(r - \frac{\sigma_S^2}{2}\right)(T - m) + \left(r - \frac{\sigma_S^2}{2}\right)\sum_{j=1}^{m}\frac{j}{m}, \tag{A.18}$$

$$\sigma_{G(u)}^2 = \sigma_S^2\left[(T - m) + \sum_{j=1}^{m}\frac{j^2}{m^2}\right]. \tag{A.19}$$

With the equation (A.13) and (A.17) on hand, the covariances would be formulated as:

$$\begin{aligned}
&\operatorname{cov}\left[\ln G_n(u), \ln G_m(S)\right] \\
&= \operatorname{cov}\left\{\sigma_u\left[\sum_{t=1}^{T-n}\Delta W_{ut}^Q + \frac{1}{n}\sum_{t=1}^{n}(n - t + 1)\Delta W_{u(T-n+t)}^Q\right],\right. \\
&\qquad \left.\sigma_S\left[\sum_{t=1}^{T-m}\Delta W_{3t}^Q + \frac{1}{m}\sum_{t=1}^{m}(m - t + 1)\Delta W_{3(T-m+t)}^Q\right]\right\} \\
&= \sigma_u\sigma_S\operatorname{cov}\left\{\left[\sum_{t=1}^{T-n}\Delta W_{ut}^Q + \frac{1}{n}\sum_{t=1}^{n}(n - t + 1)\Delta W_{u(T-n+t)}^Q\right],\right. \\
&\qquad \left.\left[\sum_{t=1}^{T-m}\Delta W_{3t}^Q + \frac{1}{m}\sum_{t=1}^{m}(m - t + 1)\Delta W_{3(T-m+t)}^Q\right]\right\}.
\end{aligned}$$

Then we could further simplify the covariance formulas under different settings of observation periods of time, say n and m.

Case 1: the same observations (usual case) ($n = m$)

$$\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right) \tag{A.20}$$

$$= \sigma_u \sigma_S \rho_{u3} \left[(T - n) + \frac{1}{6n}(n+1)(2n+1) \right]$$

By the similar way, we could derive the following formulas:

Case 2: the observations of foreign stock are more than those of domestic stock ($n > m$)

$$\begin{aligned} &\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right) \\ &= \sigma_u \sigma_S \rho_{u3} \\ &\times \left[(T - n) + \frac{1}{2n}(n+m+1)(n - m) + \frac{1}{6n}(m+1)(2m+1) \right] \end{aligned} \tag{A.21}$$

Case 3: the observations of foreign stock are less than those of domestic stock ($n < m$)

$$\begin{aligned} &\operatorname{cov}\left(\ln G_n(u), \ln G_m(S)\right) \\ &= \sigma_u \sigma_S \rho_{u3} \\ &\times \left[(T - m) + \frac{1}{2m}(n+m+1)(m - n) + \frac{1}{6m}(n+1)(2n+1) \right] \end{aligned} \tag{A.22}$$

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無套利評價方法：跨國移動平均交換選擇權

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摘要

本文提供了跨國移動平均交換選擇權的分析公式，該選擇權稱為 CCMAE 選擇權。就是一種選擇權，在一段時間內以平均價格，將一種國內資產換成另一種外國資產。實際上，它是在交易市場上常見選擇權，諸如，匯率連動、移動平均、及交換等選擇權之混合選擇權。換言之，我們提供的選擇權可以指定為標準的選擇權，就如同，匯率連動、移動平均、及交換等選擇權，為市場上標準的選擇權。接著，我們透過蒙地卡羅模擬、蒙地卡羅積分法、及數值積分方法等，驗證了 CCMAE 選擇權分析公式的準確性。

我們研究兩種類型的 CCMAE 選擇權: 其中一種，報酬指定為固定外匯匯率，其指的是固定匯率 CCMAE 選擇權 (Fixed - CCMAE 選擇權); 另一種，報酬指定為浮動外匯匯率，其指的是浮動匯率 CCMAE 選擇權 (Floating - CCMAE 選擇權)。

關鍵字：跨通貨，移動平均，交換選擇權，匯率連動選擇權

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